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A NUMERICAL METHOD FOR TWO-DIMENSIONAL LAGRANGIAN HYDRODYNAMICS

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ABSTRACT

A completely Lagrangian scheme for differencing hydrodynamical equations in two dimensions is described. The method conserves mass exactly. The advantages of Lagrangian over Eulerian schemes are briefly mentioned. An appendix gives the generalization to three dimensions.

A Numerical Method for Two-Dimensional Lagrangian Hydrodynamics

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With the increasing availability of high speed computing machines having large fast-memory storage it becomes possible to undertake the numerical investigation of hydrodynamic shock problems in two dimensions. Here is presented in outline a simple scheme for setting up the difference equations of such problems in purely Lagrangian form.

Introduce the following notation: x, y = Lagrangian coordinates, X, Y = Eulerian coordinates, U, V = velocity components, P = pressure, Q = artificial longitudinal viscous pressure[1], and G = specific volume. Then the basic hydrodynamical equations are

$$\left.\begin{aligned} \dot{U} &= -G\partial(P + Q)/\partial X \\ \dot{V} &= -G\partial(P + Q)/\partial Y \end{aligned}\right\}, \qquad (1)$$

$$\dot{X} = U, \quad \dot{Y} = V, \qquad (2)$$

$$\dot{G} = G\left[\partial U/\partial X + \partial V/\partial Y\right], \qquad (3)$$

$$d(PG^{\gamma}) = -(\gamma - 1)Q\, G^{\gamma-1}\, dG \qquad (4)$$

where the dot denotes the total (or Lagrangian) time differentiation and where we deal with a "γ-law" gas for simplicity. The viscous pressure Q will be defined presently (Eq. (11)).

In order to cast the hydrodynamical equations into completely Lagrangian form, it is necessary to take space derivatives with respect to Lagrangian coordinates. First introduce a "reference specific volume" g, related to G by

$$\frac{dxdy}{g} = \frac{dXdY}{G} \qquad (5)$$

or

$$g = G/J \tag{6}$$

where

$$J = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{vmatrix} \tag{7}$$

Next, invert the matrix of the Jacobian (7):

$$\begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \frac{\partial Y}{\partial y} & -\frac{\partial X}{\partial y} \\ -\frac{\partial Y}{\partial x} & \frac{\partial X}{\partial x} \end{pmatrix} \tag{8}$$

Expansion of the derivatives in (1) and use of Eq. (8) then leads one to the following basic set of completely "Lagrangianized" hydrodynamical equations:

$$\left.\begin{aligned} \dot{U} &= -g\left[P + Q,\ Y\right] \\ \dot{V} &= +g\left[P + Q,\ X\right] \end{aligned}\right. \tag{9}$$

$$\dot{X} = U, \quad \dot{Y} = V, \tag{2}$$

$$G = g\left[X,\ Y\right], \tag{10}$$

$$Q = \begin{cases} \left(c\Delta g^{-1}\right)^2 G^{-1}\dot{G}^2, & \text{for } \dot{G} < 0, \\ 0, & \text{for } \dot{G} \geqq 0, \end{cases} \tag{11}$$

$$d(PG^{\gamma}) = -(\gamma - 1)QG^{\gamma-1}dG, \tag{4}$$

where we have introduced the abbreviation

$$\left[A,\ B\right] \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}. \tag{12}$$

In Eq. (11), Δ is the Lagrangian mesh spacing (see below), and c is a constant near unity (see reference 1).

To set up a set of difference equations for numerical computation, a network of points may be introduced which has the topology (or connectivity) of a square mesh. If the points are labeled by integer-pairs

(k,ℓ) in a consecutive fashion, then the Lagrangian gradients may be taken as

$$(\partial A/\partial x)_{k+\frac{1}{2},\ell+\frac{1}{2}} = \tfrac{1}{2}\Delta^{-1}\left(A_{k+1,\ell+1} + A_{k+1,\ell} - A_{k,\ell+1} - A_{k,\ell}\right), \quad (13)$$

$$(\partial A/\partial y)_{k+\frac{1}{2},\ell+\frac{1}{2}} = \tfrac{1}{2}\Delta^{-1}\left(A_{k+1,\ell+1} - A_{k+1,\ell} + A_{k,\ell+1} - A_{k,\ell}\right), \text{etc.} \quad (14)$$

It is to be noted that mass is <u>exactly</u> conserved in this scheme. For if $\Omega_{k+\frac{1}{2},\ell+\frac{1}{2}}$ denotes the area of the Eulerian quadrilateral centered (in Lagrangian space) at $(k+\frac{1}{2}, \ell+\frac{1}{2})$, then the total mass of the gas should be computed at any instant as

$$M = \sum_{k,\ell} \Omega_{k+\frac{1}{2},\ell+\frac{1}{2}}\left(G_{k+\frac{1}{2},\ell+\frac{1}{2}}\right)^{-1} \quad (15)$$

But it is easy to show that

$$\Omega_{k+\frac{1}{2},\ell+\frac{1}{2}} = \Delta^2 [X,Y]_{k+\frac{1}{2},\ell+\frac{1}{2}}, \quad (16)$$

if the derivatives in the Jacobian are calculated according to Eqs. (13), (14). Hence, using (10),

$$M = \Delta^2 \sum_{k,\ell} \left(g_{k+\frac{1}{2},\ell+\frac{1}{2}}\right)^{-1} = \text{constant.} \quad (17)$$

We shall not set down the differenced form of the hydrodynamical equations here. It is sufficient to observe that the difference equations can easily be centered so as to give second order accuracy in space and time (except in a shock transition zone). The quantities, U, V, X, Y are evaluated at integral mesh points, and the quantities G, P, Q at half-odd-integral points. The time interval must be chosen so as to satisfy the stability condition[1]

$$\Delta t < \frac{\Delta}{2cg}\sqrt{\frac{G}{P}} \quad (18)$$

The advantages of a Lagrangian scheme over an Eulerian one are obvious: 1) Boundary conditions are much more easily applied. 2) Moving interfaces as well as boundaries are automatically taken care of. 3) Computations are always confined to the physical region of interest. The single disadvantage of the Lagrangian scheme is the difficulty introduced by slippages along shear planes. However, it is felt that this difficulty can easily be overcome by occasional readjusting of the mesh.

The method proposed here should be quite flexible. It is not necessary to start from a rectangular mesh at time $t = 0$. Any "freehand" net (drawn to fit surface discontinuities, for example) will do. If G^o is the initial specific volume, it is only necessary to calculate the "reference specific volume" g in advance, according to the formula

$$g = G^o/J^o \tag{19}$$

where J^o is the initial Jacobian.

Appendix:

The method outlined above can easily be extended to three dimensions. The basic hydrodynamical equations then become

$$\left.\begin{aligned} \dot{U} &= -g\left[P + Q,\ Y,\ Z\right] \\ \dot{V} &= -g\left[P + Q,\ Z,\ X\right] \\ \dot{W} &= -g\left[P + Q,\ X,\ Y\right] \end{aligned}\right\} \tag{20}$$

$$\dot{X} = U, \quad \dot{Y} = V, \quad \dot{Z} = W, \tag{21}$$

$$G = g\left[X,\ Y,\ Z\right], \tag{22}$$

where

$$\left[A,\ B,\ C\right] = \frac{\partial(A,B,C)}{\partial(x,y,z)}$$

$$= \frac{\partial A}{\partial x}\frac{\partial B}{\partial y}\frac{\partial C}{\partial z} + \frac{\partial A}{\partial y}\frac{\partial B}{\partial z}\frac{\partial C}{\partial x} + \frac{\partial A}{\partial z}\frac{\partial B}{\partial x}\frac{\partial C}{\partial y}$$

$$- \frac{\partial A}{\partial x}\frac{\partial B}{\partial y}\frac{\partial C}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial y}\frac{\partial C}{\partial x} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}\frac{\partial C}{\partial z} \tag{23}$$

Equations (4) and (11) remain unchanged.

Gradients can be computed by the obvious differencing scheme;

$$\left(\frac{\partial A}{\partial x}\right)_{k+\frac{1}{2},\ell+\frac{1}{2},m+\frac{1}{2}}$$
$$= \tfrac{1}{4}\Delta^{-1}\Big(A_{k+1,\ell+1,m+1} + A_{k+1,\ell+1,m} + A_{k+1,\ell,m+1} + A_{k+1,\ell,m}$$
$$- A_{k,\ell+1,m+1} - A_{k,\ell+1,m} - A_{k,\ell,m+1} - A_{k,\ell,m}\Big), \tag{24}$$

$$\left(\frac{\partial A}{\partial y}\right)_{k+\frac{1}{2},\ell+\frac{1}{2},m+\frac{1}{2}}$$
$$= \tfrac{1}{4}\Delta^{-1}\Big(A_{k+1,\ell+1,m+1} + A_{k+1,\ell+1,m} - A_{k+1,\ell,m+1} - A_{k+1,\ell,m}$$
$$+ A_{k,\ell+1,m+1} + A_{k,\ell+1,m} - A_{k,\ell,m+1} - A_{k,\ell,m}\Big), \tag{25}$$

$$\left(\frac{\partial A}{\partial z}\right)_{k+\frac{1}{2},\ell+\frac{1}{2},m+\frac{1}{2}}$$
$$= \tfrac{1}{4}\Delta^{-1}\Big(A_{k+1,\ell+1,m+1} - A_{k+1,\ell+1,m} + A_{k+1,\ell,m+1} - A_{k+1,\ell,m}$$
$$+ A_{k,\ell+1,m+1} - A_{k,\ell+1,m} + A_{k,\ell,m+1} - A_{k,\ell,m}\Big). \tag{26}$$

With modern computing machines it is almost feasible to consider three-dimensional problems. Such problems have, for example, a theoretical importance in the study of turbulence.

Footnotes

(1) J. von Neumann and R. D. Richtmyer, J. App. Phys. 21, 232 (1950).